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Some multiplier lacunary sequence spaces defined by a sequence of modulus functions

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Abstract. In the present paper we introduce some multiplier sequence spaces defined by a sequence of modulus functions $F = (f_k)$. We also make an effort to study some topological properties and inclusion relations between these spaces.

1 Introduction and preliminaries

A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x + y) \le f(x) + f(y)$ for all $x \ge 0$, $y \ge 0$,
3. f is increasing,
4. f is continuous from right at 0.

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It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus $f(x)$ is unbounded. Subsequently, modulus function has been discussed in [1], [2], [3], [4], [5], [20], [22], [23], [24], [26] and references therein.
Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$, for all $x \in X$,
2. $p(-x) = p(x)$, for all $x \in X$,
3. $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [29], Theorem 10.4.2, P.183).

Let w denote the set of all real sequences $x = (x_n)$. By ℓ_∞ and c, we denote respectively the Banach space of bounded and the Banach space of convergent sequences $x = (x_n)$, both normed by $\|x\| = \sup_n |x_n|$. A linear functional $\mathcal{L}$ on ℓ_∞ is said to be a Banach limit (see [6]) if it has the properties :

1. $\mathcal{L}(x) \geq 0$ if $x \geq 0$ (i.e. $x_n \geq 0$ for all n),
2. $\mathcal{L}(e) = 1$, where $e = (1, 1, \cdots)$,
3. $\mathcal{L}(Dx) = \mathcal{L}(x)$,

where the shift operator D is defined by $(Dx_n) = (x_{n+1})$.
Let $\mathfrak{B}$ be the set of all Banach limits on ℓ_∞. A sequence x is said to be almost convergent to a number L if $\mathcal{L}(x) = L$ for all $\mathcal{L} \in \mathfrak{B}$. Lorentz [17] has shown that x is almost convergent to L if and only if

$$t_{km} = t_{km}(x) = \frac{x_m + x_{m+1} + \cdots + x_{m+k}}{k+1} \to L \text{ as } k \to \infty, \text{ uniformly in } m.$$

Let $\hat{c}$ denote the set of all almost convergent sequences. Maddox [18] and (independently) Freedman et al. [13] have defined x to be strongly almost convergent to a number L if

$$\frac{1}{k+1}\sum_{i=0}^{k}|x_{i+m}-L| \to 0 \text{ as } k\to\infty, \text{ uniformly in } m.$$

Let $[\hat{c}]$ denote the set of all strongly almost convergent sequences. It is easy to see that $[\hat{c}] \subset \hat{c} \subset \ell_\infty$. Das and Sahoo [11] defined the sequence space

$$[w(p)] = \left\{x \in w : \frac{1}{n+1}\sum_{k=0}^{n}|t_{km}(x-L)|^{p_k} \to 0 \text{ as } n\to\infty, \text{ uniformly in } m.\right\}$$

and investigated some of its properties.

The space of lacunary strong convergence have been introduced by Freedman et al. [13]. A sequence of positive integers $\theta = (k_r)$ is called "lacunary" if $k_0 = 0, 0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ is defined by Freedman et al. [13] as follows:

$$N_\theta = \left\{x = (x_i) : \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}|x_i - s| = 0, \text{ for some } s\right\}.$$

Lacunary sequence spaces were studied by many authors (see [7], [8], [9]) and references therein.

Let $F = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be a bounded sequence of positive real numbers, $u = (u_k)$ be any sequence of strictly positive real numbers and X be a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q. By $w(X)$ we denote the space of all sequences $x = (x_k)$ for all k. In the present paper we define the following classes of sequences:

$$(w,\theta,F,u,p,q) = \left\{x = (x_k) \in w(X) : \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}u_k\Big[f_k(q(t_{km}(x-L)))\Big]^{p_k} = 0, \right.$$
$$\left. \text{uniformly in } m, \quad \text{for some } L\right\},$$

$$(w,\theta,F,u,p,q)_0 = \left\{x = (x_k) \in w(X) : \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}u_k\Big[f_k(q(t_{km}(x)))\Big]^{p_k} = 0, \right.$$
$$\left. \text{uniformly in } m\right\}$$

and

$$(w,\theta,F,u,p,q)_\infty=\Big\{x=(x_k)\in w(X):\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r}u_k\Big[f_k(q(t_{km}(x)))\Big]^{p_k}<\infty\Big\}.$$

If we take $f(x)=x$, we have

$$(w,\theta,u,p,q)=\Big\{x=(x_k)\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}u_k\Big[(q(t_{km}(x-L)))\Big]^{p_k}=0,$$

$$\text{uniformly in } m,\quad \text{for some } L\Big\},$$

$$(w,\theta,u,p,q)_0=\Big\{x=(x_k)\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}u_k\Big[(q(t_{km}(x)))\Big]^{p_k}=0,$$

$$\text{uniformly in } m\Big\}$$

and

$$(w,\theta,u,p,q)_\infty=\Big\{x=(x_k)\in w(X):\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r}u_k\Big[(q(t_{km}(x)))\Big]^{p_k}<\infty\Big\}.$$

If we take $p=(p_k)=1$ for all $k\in\mathbb{N}$, we have

$$(w,\theta,F,u,q)=\Big\{x=(x_k)\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}u_k\Big[f_k(q(t_{km}(x-L)))\Big]=0,$$

$$\text{uniformly in } m,\quad \text{for some } L\Big\},$$

$$(w,\theta,F,u,q)_0=\Big\{x=(x_k)\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}u_k\Big[f_k(q(t_{km}(x)))\Big]=0,$$

$$\text{uniformly in } m\Big\}$$

and

$$(w,\theta,F,u,q)_\infty=\Big\{x=(x_k)\in w(X):\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r}u_k\Big[f_k(q(t_{km}(x)))\Big]<\infty\Big\}.$$

If we take $f(x)=x$ and $u=(u_k)=1$ for all $k\in\mathbb{N}$, we have

$$(w,\theta,p,q)=\Big\{x=(x_k)\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[(q(t_{km}(x-L)))\Big]^{p_k}=0,$$

$$\text{uniformly in } m, \quad \text{for some } L\Big\},$$

$$(w,\theta,p,q)_0=\Big\{x=(x_k)\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[(q(t_{km}(x)))\Big]^{p_k}=0,$$

$$\text{uniformly in } m\Big\}$$

and

$$(w,\theta,p,q)_\infty=\Big\{x=(x_k)\in w(X):\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r}\Big[(q(t_{km}(x)))\Big]^{p_k}<\infty\Big\}.$$

If we take $\theta=(2^r)$, then the spaces (w,θ,F,u,p,q), $(w,\theta,F,u,p,q)_0$ and $(w,\theta,F,u,p,q)_\infty$ reduces to (w,F,u,p,q), $(w,F,u,p,q)_0$ and $(w,F,u,p,q)_\infty$. Throughout the paper Z will denote the $0,1$ or ∞. The following inequality will be used throughout the paper. If $0<h=\inf p_k\le p_k\le\sup p_k=H$, $D=\max(1,2^{H-1})$ then

$$|a_k+b_k|^{p_k}\le D\{|a_k|^{p_k}+|b_k|^{p_k}\} \tag{1}$$

for all k and $a_k,b_k\in\mathbb{C}$. Also $|a|^{p_k}\le\max(1,|a|^H)$ for all $a\in\mathbb{C}$.

In this paper we study some topological properties and prove some inclusion relations between above defined classes of sequences.

2 Main results

Theorem 1 *Let* $F=(f_k)$ *be a sequence of modulus functions,* $u=(u_k)$ *be any sequence of strictly positive real numbers and* $p=(p_k)$ *be a bounded sequence of positive real numbers. Then* $(w,\theta,F,u,p,q)_Z$ *are linear spaces over the field of complex numbers* $\mathbb{C}$.

Proof. We shall prove the result for $Z=0$. Let $x=(x_k),y=(y_k)\in(w,\theta,F,u,p,q)_0$ and $\alpha,\beta\in\mathbb{C}$. Then there exist integers M_α and N_β such that $|\alpha|\le M_\alpha$ and $|\beta|\le N_\beta$. By using inequality (1.1) and the properties of modulus function, we have

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r}u_k[f_k(q(t_{km}(\alpha x_k+\beta y_k)))]^{p_k} &\le \frac{1}{h_r}\sum_{k\in I_r}u_k[f_k(q(\alpha t_{km}x_k+\beta t_{km}y_k))]^{p_k}\\ &\le D\frac{1}{h_r}\sum_{k\in I_r}u_k[M_\alpha f_k(q(t_{km}(x_k)))]^{p_k}\end{aligned}$$

$$
\begin{aligned}
&+ \quad D\frac{1}{h_r}\sum_{k\in I_r} u_k[N_\beta f_k(q(t_{km}(y_k)))]^{p_k} \\
&\leq \quad DM_\alpha^H \frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x_k)))]^{p_k} \\
&+ \quad DN_\beta^H \frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(y_k)))]^{p_k} \\
&\to \quad 0, \text{ uniformly in } m.
\end{aligned}
$$

This proves that $(w, \theta, F, u, p, q)_0$ is a linear space. Similarly, we can prove that (w, θ, F, u, p, q) and $(w, \theta, F, u, p, q)_\infty$ are linear spaces. □

Theorem 2 *Let* $F = (f_k)$ *be a sequence of modulus functions. Then we have*

$$(w, \theta, F, u, p, q)_0 \subset (w, \theta, F, u, p, q) \subset (w, \theta, F, u, p, q)_\infty.$$

Proof. The inclusion $(w, \theta, F, u, p, q)_0 \subset (w, \theta, F, u, p, q)$ is obvious. Now, let $x = (x_k) \in (w, \theta, F, u, p, q)$ then

$$\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x_k)))]^{p_k} \to 0, \text{ uniformly in } m.$$

Now by using (1.1) and the properties of modulus function, we have

$$
\begin{aligned}
\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x_k)))]^{p_k} &= \sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x_k - L + L)))]^{p_k} \\
&\leq D\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x_k - L)))]^{p_k} \\
&+ D\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(L))]^{p_k} \\
&\leq D\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r} [f_k(q(t_{km}(x_k - L)))]^{p_k} \\
&+ D\max\{f_k(q(L))^h, f_k(q(L))^H\} \\
&< \infty.
\end{aligned}
$$

Hence $x = (x_k) \in (w, \theta, F, u, p, q)_\infty$. This proves that

$$(w, \theta, F, u, p, q) \subset (w, \theta, F, u, p, q)_\infty.$$

□

Theorem 3 *Let* $p = (p_k)$ *be a bounded sequence of positive real numbers. Then the space* $(w, \theta, F, u, p, q)_0$ *is a paranormed space with the paranorm defined by*

$$g(x) = \sup_{r,m} \left(\frac{1}{h_r} \sum_{k \in I_r} u_k [f_k(q(t_{km}(x_k)))]^{p_k} \right)^{\frac{1}{M}},$$

where $M = \max(1, \sup_k p_k)$.

Proof. Clearly $g(-x) = g(x)$. It is trivial that $t_{km} x_k = 0$ for $x = 0$. Hence we get $g(0) = 0$. Since $\frac{p_k}{M} \leq 1$ and $M \geq 1$, using the Minkowski's inequality and definition of modulus function, for each x, we have

$$\left(\frac{1}{h_r} \sum_{k \in I_r} u_k [f_k(q(t_{km}(x_k + y_k)))]^{p_k} \right)^{\frac{1}{M}}$$

$$\begin{aligned} &\leq \left(\frac{1}{h_r} \sum_{k \in I_r} u_k [f_k(q(t_{km}(x_k))) + f_k(q(t_{km}(y_k)))]^{p_k} \right)^{\frac{1}{M}} \\ &\leq \left(\frac{1}{h_r} \sum_{k \in I_r} u_k [f_k(q(t_{km}(x_k)))]^{p_k} \right)^{\frac{1}{M}} + \left(\frac{1}{h_r} \sum_{k \in I_r} u_k [f_k(q(t_{km}(y_k)))]^{p_k} \right)^{\frac{1}{M}}. \end{aligned}$$

Now it follows that g is subadditive. Finally, to check the continuity of scalar multiplication, let us take any complex number λ. By definition of modulus function F, we have

$$\begin{aligned} g(\lambda x) &= \sup_{r,m} \left(\frac{1}{h_r} \sum_{k \in I_r} u_k [f_k(q(t_{km}(\lambda x_k)))]^{p_k} \right)^{\frac{1}{M}} \\ &\leq K^{\frac{H}{M}} g(x), \end{aligned}$$

where $K = 1 + [|\lambda|]$ ($[|\lambda|]$ denotes the integer part of λ). Since F is a sequence of modulus functions, we have $x \to 0$ implies $g(\lambda x) \to 0$. Similarly $x \to 0$ and $\lambda \to 0$ implies $g(\lambda x) \to 0$. Finally, we have for fixed x and $\lambda \to 0$ implies $g(\lambda x) \to 0$. □

Theorem 4 *Let* F' *and* F'' *be any two sequences of modulus functions. For any bounded sequences* $p = (p_k)$ *and* $t = (t_k)$ *of strictly positive real numbers and for any two sequences of seminorms* q_1 *and* q_2, *we have*

(i) $(w, \theta, F', u, p, q)_Z \subset (w, \theta, F' o F'', u, p, q)_Z$;

(ii) $(w, \theta, F', u, p, q)_Z \cap (w, \theta, F'', u, p, q)_Z \subset (w, \theta, F' + F'', u, p, q)_Z$;

(iii) $(w,\theta,F,u,p,q_1)_Z \cap (w,\theta,F,u,p,q_2)_Z \subset (w,\theta,F,u,p,q_1+q_2)_Z$;
(iv) If q_1 *is stronger than* q_2 *then* $(w,\theta,F,u,p,q_1)_Z \subset (w,\theta,F,u,p,q_2)_Z$;
(v) If q_1 *equivalent to* q_2 *then* $(w,\theta,F,u,p,q_1)_Z = (w,\theta,F,u,p,q_2)_Z$;
(vi) $(w,\theta,F,u,p,q)_Z \cap (w,\theta,F,u,t,q)_Z \neq \phi$.

Proof. It is easy to prove so we omit the details. □

Corollary 2.5. *Let* $F = (f_k)$ *be a sequence of modulus functions. Then* $(w,\theta,u,q)_Z \subset (w,\theta,F,u,q)_Z$.

Proof. Let $x = (x_k) \in (w,\theta,u,q)_Z$ and $\epsilon > 0$. We can choose $0 < \delta < 1$ such that $f_k(t) < \epsilon$ for every $t \in [0,\infty)$ with $0 \leq t \leq \delta$. Then, we can write

$$\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x_k - L)))]$$

$$\begin{aligned}
&= \frac{1}{h_r}\sum_{k\in I_r, t_{km}(x_k-L)\leq\delta} u_k[f_k(q(t_{km}(x_k - L)))] \\
&+ \frac{1}{h_r}\sum_{k\in I_r, t_{km}(x_k-L)>\delta} u_k[f_k(q(t_{km}(x_k - L)))] \\
&\leq \max\left\{f_k(\epsilon), f_k(\epsilon)\right\} \\
&+ \max\left\{1, (2f_k(1)\delta^{-1})\right\}\frac{1}{h_r}\sum_{k\in I_r, t_{km}(x_k-L)>\delta} u_k[f_k(q(t_{km}(x_k - L)))].
\end{aligned}$$

Therefore $x = (x_k) \in (w,\theta,F,u,q)_Z$. This completes the proof of the theorem. Similarly, we can prove the other cases. □

Theorem 5 *Let* $F = (f_k)$ *be a sequence of modulus functions, if* $\lim_{t\to\infty}\frac{f(t)}{t} = \beta > 0$, *then* $(w,\theta,u,q)_Z = (w,\theta,F,u,q)_Z$.

Proof. By Corollary 2.5, we need only to show that $(w,\theta,F,u,q)_Z \subset (w,\theta,u,q)_Z$. Let $\beta > 0$ and $x \in (w,\theta,F,u,q)_Z$. Since $\beta > 0$, we have $f(t) \geq \beta t$ for all $t \geq 0$. Hence

$$\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x_k - L)))] \geq \frac{\beta}{h_r}\sum_{k\in I_r} u_k(q(t_{km}(x_k - L))).$$

Therefore, $x \in (w,\theta,u,q)_Z$. □

Theorem 6 *Let* $0 < p_k \leq t_k$ *and* $\left(\frac{t_k}{p_k}\right)$ *be bounded. Then*

$$(w, \theta, F, u, t, q)_Z \subset (w, \theta, F, u, p, q)_Z.$$

Proof. Let $x = (x_k) \in (w, \theta, F, u, t, q)_Z$. Let $r_k = u_k[f_k(q(t_{km}x_k - L))]^{t_k}$ and $\lambda_k = (\frac{p_k}{t_k})$ for all $k \in \mathbb{N}$ so that $0 < \lambda \leq \lambda_k \leq 1$. Define the sequences (u_k) and (v_k) as follows:
For $r_k \geq 1$, let $u_k = r_k$ and $v_k = 0$ and for $r_k < 1$, let $u_k = 0$ and $v_k = r_k$. Then clearly for all $k \in \mathbb{N}$, we have $r_k = u_k + v_k$, $r_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$, $u_k^{\lambda_k} \leq u_k \leq r_k$ and $v_k^{\lambda_k} = v_k^{\lambda}$. Therefore

$$\frac{1}{h_r}\sum_{k\in I_r} r_k^{\lambda_k} \leq \frac{1}{h_r}\sum_{k\in I_r} r_k + [\frac{1}{h_r}\sum_{k\in I_r} v_k]^{\lambda}.$$

Hence $x = (x_k) \in (w, \theta, F, u, p, q)_Z$. Thus $(w, \theta, F, u, t, q)_Z \subset (w, \theta, F, u, p, q)_Z$.
□

Theorem 7 *Let* $\theta = (k_r)$ *be a lacunary sequence. If* $1 < \liminf_r q_r \leq \lim \sup_r q_r < \infty$ *then for any modulus function* F, *we have* $(w, F, u, p, q)_0 = (w, \theta, F, u, p, q)_0$.

Proof. Suppose $\liminf_r q_r > 1$ then there exist $\delta > 0$ such that $q_r = (\frac{k_r}{k_{r-1}}) \geq 1 + \delta$ for all $r \geq 1$. Then for $x = (x_k) \in (w, F, u, p, q)_0$, we write

$$\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x)))]^{p_k}$$

$$= \frac{1}{h_r}\sum_{k=1}^{k_r} u_k[f_k(q(t_{km}(x)))]^{p_k} - \frac{1}{h_r}\sum_{k=1}^{k_{r-1}} u_k[f_k(q(t_{km}(x)))]^{p_k}$$

$$= \frac{k_r}{h_r}\Big(k_r^{-1}\sum_{k=1}^{k_r} u_k[f_k(q(t_{km}(x)))]^{p_k}\Big) - \frac{k_{r-1}}{h_r}\Big(k_{r-1}^{-1}\sum_{k=1}^{k_{r-1}} u_k[f_k(q(t_{km}(x)))]^{p_k}\Big).$$

Since $h_r = k_r - k_{r-1}$, we have $\frac{k_r}{h_r} \leq \frac{1+\delta}{\delta}$ and $\frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}$. The terms

$$k_r^{-1}\sum_{k=1}^{k_r} u_k[f_k(q(t_{km}(x)))]^{p_k}$$

and $\frac{k_{r-1}}{h_r}\left(k_{r-1}^{-1}\sum_{k=1}^{k_{r-1}} u_k[f_k(q(t_{km}(x)))]^{p_k}\right)$ both converge to zero, uniformly in m and it follows that

$$\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x)))]^{p_k} \to 0,$$

as $r \to \infty$ uniformly in m, that is, $x \in (w, \theta, F, u, p, q)_0$.
If $\limsup_r q_r < \infty$, there exists $B > 0$ such that $\eta_r < B$ for all $r \geq 1$. Let $x \in (w, \theta, F, u, p, q)_0$ and $\epsilon > 0$ be given. Then there exists $R > 0$ such that for every $j \geq R$ and all m.

$$A_j = \frac{1}{h_j}\sum_{k\in I_j} u_k[f_k(q(t_{km}(x)))]^{p_k} < \epsilon.$$

We can also find $K > 0$ such that $A_j < K$ for all $j = 1, 2, \cdots$. Now let t be any integer with $k_{r-1} < t \leq k_r$, where $r > R$. Then

$$
\begin{aligned}
&t^{-1}\textstyle\sum_{k=1}^{t} u_k[f_k(q(t_{km}(x)))]^{p_k}\\
&\leq \quad k_{r-1}^{-1}\sum_{k=1}^{k_r} u_k[f_k(q(t_{km}(x)))]^{p_k}\\
&= \quad k_{r-1}^{-1}\sum_{k\in I_1} u_k[f_k(q(t_{km}(x)))]^{p_k} + k_{r-1}^{-1}\sum_{k\in I_2} u_k[f_k(q(t_{km}(x)))]^{p_k}\\
&+ \quad \cdots + k_{r-1}^{-1}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x)))]^{p_k}\\
&= \quad \frac{k_1}{k_{r-1}}k_1^{-1}\sum_{k\in I_1} u_k[f_k(q(t_{km}(x)))]^{p_k} + \frac{k_2-k_1}{k_{r-1}}(k_2-k_1)^{-1}\sum_{k\in I_2} u_k[f_k(q(t_{km}(x)))]^{p_k}\\
&+ \quad \cdots + \frac{k_R-k_{R-1}}{k_{r-1}}(k_R-k_{R-1})^{-1}\sum_{k\in I_R} u_k[f_k(q(t_{km}(x)))]^{p_k}\\
&+ \quad \cdots + \frac{k_r-k_{r-1}}{k_{r-1}}(k_r-k_{r-1})^{-1}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x)))]^{p_k}\\
&= \quad \frac{k_1}{k_{r-1}}A_1 + \frac{k_2-k_1}{k_{r-1}}A_2 + \cdots + \frac{k_R-k_{R-1}}{k_{r-1}}A_R\\
&+ \quad \frac{k_{R+1}-k_R}{k_{r-1}}A_{R+1} + \cdots + \frac{k_r-k_{r-1}}{k_{r-1}}A_r
\end{aligned}
$$

$$\leq (\sup_{j\geq 1} A_j)\frac{k_R}{k_{r-1}} + (\sup_{j\geq R} A_j)\frac{k_r - k_R}{k_{r-1}}$$
$$\leq K\frac{k_R}{k_{r-1}} + \epsilon B.$$

Since $K_{r-1} \to \infty$ as $t \to \infty$, it follows that $t^{-1}\sum_{k=1}^{t} u_k[f_k(q(t_{km}(x)))]^{p_k} \to 0$ uniformly in m and consequently $x \in (w, F, u, p, q)_0$. □

3 Statistical convergence

The notion of statistical convergence was introduced by Fast [12] and Schoenberg [28] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy [14], Connor [10], Salat [25], Murasaleen [21], Isik [15], Savas [27], Malkosky and Savas [20], Kolk [16], Maddox [18, 19] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Cech compactification of natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability. The notion depends on the density of subsets of the set $\mathbb{N}$ of natural numbers.

A subset E of $\mathbb{N}$ is said to have the natural density $\delta(E)$ if the following limit exists: $\delta(E) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_E(k)$, where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c) = 1 - \delta(E)$.

Let θ be a lacunary sequence, then the sequence $x = (x_k)$ is said to be q-lacunary almost statistically convergent to the number L provided that for every $\epsilon > 0$,

$$\lim_{r\to\infty}\frac{1}{h_r}|\{k \in I_r : q(t_{km}(x - L)) \geq \epsilon\}| = 0, \text{ uniformly in } m.$$

In this case we write $[S_\theta]_q - \lim x = L$ or $x_k \to L([S_\theta]_q)$ and we define

$$[S_\theta]_q = \Big\{x \in w(X) : [S_\theta]_q - \lim x = L, \text{ for some } L\Big\}.$$

In the case $\theta = (2^r)$, we shall write $[S]_q$ instead of $[S_\theta]_q$.

Theorem 8 *Let* $F = (f_k)$ *be a sequence of modulus functions and* $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. *Then* $(w, \theta, F, u, p, q) \subset [S_\theta]_q$.

Proof. Let $x \in (w, \theta, F, u, p, q)$ and $\epsilon > 0$ be given. Then

$$\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x-L)))]^{p_k}$$

$$\begin{aligned}
&\geq \frac{1}{h_r}\sum_{k\in I_r, q(t_{km}(x-L))\geq\epsilon} u_k[f_k(q(t_{km}(x-L)))]^{p_k}\\
&\geq \frac{1}{h_r}\sum_{k\in I_r, q(t_{km}(x-L))\geq\epsilon} u_k[f_k(\epsilon)]^{p_k}\\
&\geq \frac{1}{h_r}\sum_{k\in I_r, q(t_{km}(x-L))\geq\epsilon} \min(u_k[f_k(\epsilon)]^h, u_k[f_k(\epsilon)]^H)\\
&\geq \frac{1}{h_r}|\{k \in I_r : q(t_{km}(x-L)) \geq \epsilon\}| \min(u_k[f_k(\epsilon)]^h, u_k[f_k(\epsilon)]^H).
\end{aligned}$$

Hence $x \in [S_\theta]_q$. □

Theorem 9 *Let* $F = (f_k)$ *be a bounded sequence of modulus functions and* $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. *Then* $[S_\theta]_q \subset (w, \theta, F, u, p, q)$.

Proof. Suppose that $F = f_k$ is bounded. Then there exists an integer K such that $f_k(t) < K$, for all $t \geq 0$. Then

$$\begin{aligned}
\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x-L)))]^{p_k} &= \frac{1}{h_r}\sum_{k\in I_r, q(t_{km}(x-L))\geq\epsilon} u_k[f_k(q(t_{km}(x-L)))]^{p_k}\\
&+ \frac{1}{h_r}\sum_{k\in I_r, q(t_{km}(x-L))<\epsilon} u_k[f_k(q(t_{km}(x-L)))]^{p_k}\\
&\leq \frac{1}{h_r}\sum_{k\in I_r, q(t_{km}(x-L))\geq\epsilon} \max(K^h, K^H)\\
&+ \frac{1}{h_r}\sum_{k\in I_r, q(t_{km}(x-L))<\epsilon} u_k[f_k(\epsilon)]^{p_k}\\
&\leq \max(K^h, K^H)\frac{1}{h_r}|\{k \in I_r : q(t_{km}(x-L)) \geq \epsilon\}|\\
&+ \max(u_k[f_k(\epsilon)]^h, u_k[f_k(\epsilon)]^H).
\end{aligned}$$

Hence $x \in (w, \theta, F, u, p, q)$. □

Theorem 10 $[S_\theta]_q = (w, \theta, F, u, p, q)$ *if and only if* $F = (f_k)$ *is bounded.*

Proof. Let $F = (f_k)$ be bounded. By the Theorem 3.1 and Theorem 3.2, we have

$$[S_\theta]_q = (w, \theta, F, u, p, q).$$

Conversely, suppose that F is unbounded. Then there exists a positive sequence (t_n) with $f(t_n) = n^2, n = 1, 2, \cdots$.
If we choose

$$x_k = \begin{cases} t_n, & k = n^2, n = 1, 2, \cdots, \\ 0, & \text{otherwise} \end{cases}.$$

Then we have

$$\frac{1}{n}|\{k \leq n : |x_k| \geq \epsilon\}| \leq \frac{\sqrt{n}}{n} \to 0, n \to \infty.$$

Hence $x_k \to 0([S_\theta]q)$ for $t_{0m}(x) = x_m, \theta = (2^r)$ and $q(x) = |x|$, but $x \notin (w, \theta, F, u, p, q)$. This contradicts to $[S_\theta]_q = (w, \theta, F, u, p, q)$. □

References

[1] Y. Altin, M. Et, Generalized difference sequence spaces defined by a modulus function in a locally convex space, *Soochow J. Math.*, **31** (2005), 233–243.

[2] H. Altinok, Y. Altin, M. Isik, The sequence space $Bv_\sigma(M, p, q, s)$ on seminormed spaces, *Indian J. Pure Appl. Math.*, **39**, 1 (2008), 49–58.

[3] Y. Altin, Properties of some sets of sequences defined by a modulus function, *Acta Math. Sci. Ser. B Engl.* Ed., **29**, 2 (2009), 427–434.

[4] Y. Altin, H. Altinok, R. Çolak, On some seminormed sequence spaces defined by a modulus function, *Kragujevac J. Math.*, **29** (2006), 121–132.

[5] Y. Altin, A. Gökhan, H. Altinok, Properties of some new seminormed sequence spaces defined by a modulus function, *Studia Univ. Babeş-Bolyai Math.* **L.**, 3 (2005), 13–19.

[6] S. Banach, *Theorie operations linearies*, Chelsea Publishing Co., New York (1955).

[7] T. Bilgin, Lacunary strong A-convergence with respect to a modulus, *Studia Univ. Babeş-Bolyai Math.*, **XLV**I (2001), 39–46.

[8] T. Bilgin, Lacunary strong A-convergence with respect to a sequence of modulus functions, *Appl. Math. Comput.*, **151** (2004), 595–600.

[9] R. Colak, B. C. Tripathy, M. Et, Lacunary strongly summable sequences and q-lacunary almost statistical convergence, *Vietnam J. Math.*, **34** (2006), 129–138.

[10] J. S. Connor, A topological and functional analytic approach to statistical convergence, *Appl. Numer. Harmonic Anal.*, (1999), 403–413.

[11] G. Das, S. K. Sahoo, On some sequence spaces, *J. Math. Anal. Appl.*, **164** (1992), 381–398.

[12] H. Fast, Sur la convergence statistique, *Colloq. Math.*, **2** (1951), 241–244.

[13] A. R. Freedman, J. J. Sember, M. Raphael, Some Cesaro-type summability spaces, *Proc. London Math. Soc.*, **37** (1978), 508–520.

[14] J. A. Fridy, On the statistical convergence, *Analysis*, **5** (1985), 301–303.

[15] M. Isik, On statistical convergence of generalized difference sequence spaces, *Soochow J. Math.*, **30** (2004), 197–205.

[16] E. Kolk, The statistical convergence in Banach spaces, *Acta. Comment. Univ. Tartu*, **928** (1991), 41–52.

[17] G. G. Lorentz, A contribution to the theory of divergent series, *Acta Math.*, **80** (1948), 167–190.

[18] I. J. Maddox, A new type of convergence, *Math. Proc. Cambridge Phil. Soc.*, **83** (1978), 61–64.

[19] I. J. Maddox, Statistical convergence in a locally convex space, *Math. Proc. Cambridge Phil. Soc.*, **104** (1988), 141–145.

[20] E. Malkowsky, E. Savas, Some λ-sequence spaces defined by a modulus, *Arch. Math.*, **36** (2000), 219–228.

[21] M. Mursaleen, λ-statistical convergence, *Math. Slovaca*, **50** (2000), 111–115.

[22] K. Raj, S. K. Sharma, A. K. Sharma, Some new sequence spaces defined by a sequence of modulus functions in n-normed spaces, *Int. J. Math. Sci. Eng. Appl.*, **5**, 2 (2011), 395–403.

[23] K. Raj, S. K. Sharma, Difference sequence spaces defined by sequence of modulus functions, *Proyecciones*, **30** (2011), 189–199.

[24] K. Raj, S. K. Sharma, Some difference sequence spaces defined by sequence of modulus functions, *Int. Journal of Mathematical Archive*, **2** (2011), 236–240.

[25] T. Salat, On statistical convergent sequences of real numbers, *Math. Slovaca*, **30** (1980), 139–150.

[26] E. Savas, On some generalized sequence spaces defined by a modulus, *Indian J. Pure Appl. Math.*, **30** (1999), 459–464.

[27] E. Savas, Strong almost convergence and almost λ-statistical convergence, *Hokkaido Math. J.*, **29** (2000), 531–566.

[28] I. J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly*, **66** (1959), 361–375.

[29] A. Wilansky, Summability through functional analysis, *North-Holland Math. Stud.*, Amsterdam (1984).

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